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DECEMBER, 1952

DISCUSSION OF DEFLECTIONS IN GRIDWORKS AND SLABS

(*Published in September, 1951*)

By Jack R. Benjamin, Alexander C. Scordelis, Paul J. Brennan, Serge Leliavsky Bey, W. Watters Pagon, Phil M. Ferguson, Robert V. Whitman, and Walter W. Ewell, Shigeo Okubo, and Joel I. Abrams

STRUCTURAL DIVISION

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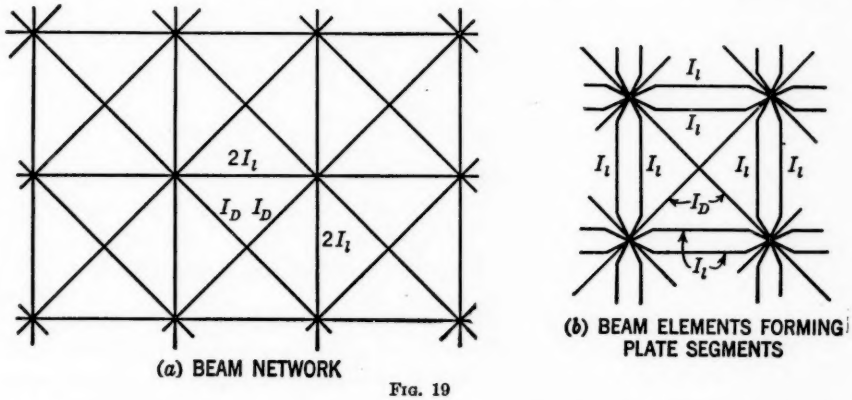
DISCUSSION

JACK R. BENJAMIN,[13] A. M. ASCE.—For a considerable time the writer has been working with the problem of a plate loaded in its own plane rather than transverse to that plane. The two problems are very similar and can be solved by the same general procedures. The authors replace the slab or plate by intersecting beams having both bending and torsional resistance. Some of the fundamental plate characteristics become lost when this is done and the evaluation of the elastic properties of the network members is then only approximate. The solution obtained by the authors shows the errors to be minor in character for the particular problem solved.

The more exact network of beams can be determined readily from the fundamental properties of a finite piece of plate, thus yielding the equivalent beam network shown in Fig. 19. This network differs from that in the paper



(a) BEAM NETWORK

(b) BEAM ELEMENTS FORMING PLATE SEGMENTS

FIG. 19

in two respects—(1) only bending stiffness is considered, and (2) diagonal intersecting beams are added. The elastic properties of the beam network members are as follows:

$$I_l = \frac{I}{2(1-\mu)} \left(\text{in which } I = \frac{l\,h^3}{1\,2} \right) \ldots\ldots\ldots\ldots\ldots (20a)$$

$$I_D = \frac{I\,\mu\,\sqrt{2}}{1-\mu^2} \text{ (for pure bending)} \ldots\ldots\ldots\ldots\ldots (20b)$$

and

$$I_D = \frac{I\,\sqrt{2}}{2(1+\mu)} \text{ (for warping)} \ldots\ldots\ldots\ldots\ldots (20c)$$

If μ is 0.333, Eqs. 20 reduce to

$$I_l = \frac{3}{8} I \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21a)$$

NOTE.—This paper by Walter W. Ewell, Shigeo Okubo, and Joel I. Abrams was published in September, 1951, as *Proceedings-Separate No. 89*. The numbering of footnotes, illustrations, tables, and equations in this Separate is a continuation of the consecutive numbering used in the original paper.

[13] Associate Prof. of Structural Eng., Stanford Univ., Stanford, Calif.

and

$$I_D = \frac{3}{8}\sqrt{2}\, I \text{ (for both bending and warping)} \ldots\ldots\ldots (21b)$$

The entire plate is then formed by the consideration of all the beam elements (Fig. 19). The side elements add stiffnesses to become $0.75\, I$. The entire plate then has been replaced by a finite number of segments, each with sides of length l. The plate deflections will then agree exactly with the beam network deflections at all beam intersections if a large number of segments are used. Errors have been found to be very small when only a very few segments are used.

The solution of the beam network requires considerable work and if extreme care is not taken in the selection and adaptation of procedures a very difficult convergence problem arises. Generally a numerical iterative solution is used and, although errors at any given point may appear small, they may be cumulative at one or more points, resulting in large local errors.

The beam network shown in Fig. 19 is exactly the same as that used for a plate loaded in its own plane if the moment of inertia is replaced by the area of the segment. This discussion shows, therefore, that pure bending and warping of a plate transversely loaded are analogus to direct stress and shear in a plate loaded in its own plane.

ALEXANDER C. SCORDELIS,[14] J. M. ASCE.—The extension of the moment-distribution method to the solving of deflections in gridworks and slabs, presented by the authors, is to be commended. Since the moment-distribution method is familiar to almost all structural engineers, the technique developed by the authors can easily be followed and understood and, in most cases, readily applied.

The solution of any problem involving the deflections in gridworks and slabs is a tedious procedure. Using the method presented in this paper, it can be seen that for gridworks that lack symmetry and have many joints the work involved becomes appreciable. A separate solution must be made for each joint displacement, then for each of these solutions reactions or holding forces must be calculated at each joint, and, finally, to obtain the deflections a set of simultaneous equations which satisfy shear relations must be solved.

In comparing this method with the moment-distribution method used in sidesway problems, an alternate procedure suggests itself. Sidesway problems—or better yet, problems involving joint dispacements for frames loaded in the plane of the structure—may be solved in one of two ways: The first is the so-called superposition method, which is analogous in many respects to that used by the authors in solving their problem; and the second is that of successive shear corrections. This latter procedure is considered by many to be superior to the superposition method since, when the number of possible joint displacements becomes great, it eliminates the necessity for making separate solutions and solving simultaneous equations.

The method of successive shear corrections can also be used in solving for deflection in gridworks and slabs. As a general case, consider the gridwork

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shown in Fig. 20, with a load of 1,000 P, at joint 1. This gridwork is the same as that shown in Fig. 5. For the solution of deflections by successive shear corrections, the general procedure can be described in five steps.

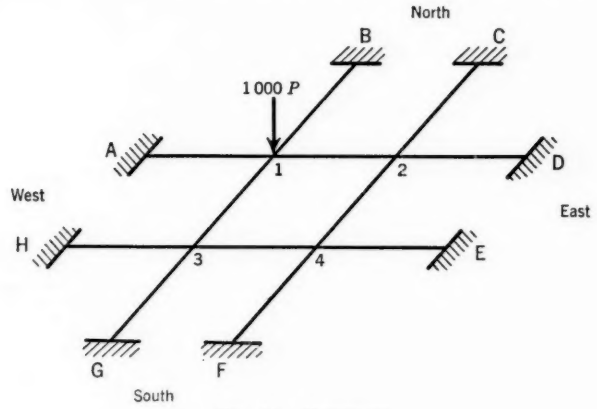


FIG. 20.—GRIDWORK

Step 1.—One shear equation is written for each possible joint displacement. These equations will involve the external load at the joint and the shears at the far ends of the members framing into the joint. These shears can be expressed in terms of the moments at each end of the four members. For example, at joint 1:

$$\left(\frac{M_{A1}+M_{1A}}{L}\right)-\left(\frac{M_{12}+M_{21}}{L}\right)+\left(\frac{M_{B1}+M_{1B}}{L}\right)-\left(\frac{M_{13}+M_{31}}{L}\right)-1{,}000\,P=0\ldots(22)$$

or simplifying, at joint 1:

$$M_{A1}+M_{1A}-M_{12}-M_{21}+M_{B1}+M_{1B}-M_{13}-M_{31}-1{,}000\,P\,L=0\ldots(23a)$$

Similarly at joint 2—

$$M_{12}+M_{21}-M_{2D}-M_{D2}+M_{C2}+M_{2C}-M_{24}-M_{42}=0\ldots(23b)$$

at joint 3—

$$M_{H3}+M_{3H}-M_{34}-M_{43}+M_{13}+M_{31}-M_{3G}-M_{G3}=0\ldots(23c)$$

and, at joint 4—

$$M_{34}+M_{43}-M_{4E}-M_{E4}+M_{24}+M_{42}-M_{4F}-M_{F4}=0\ldots(23d)$$

The signs of the moments in Eqs. 23 are in accordance with the moment-distribution convention when viewed from the south and the west.

Step 2.—With all joints locked each joint is displaced. These displacements produce fixed-end moments in the members framing into each joint.

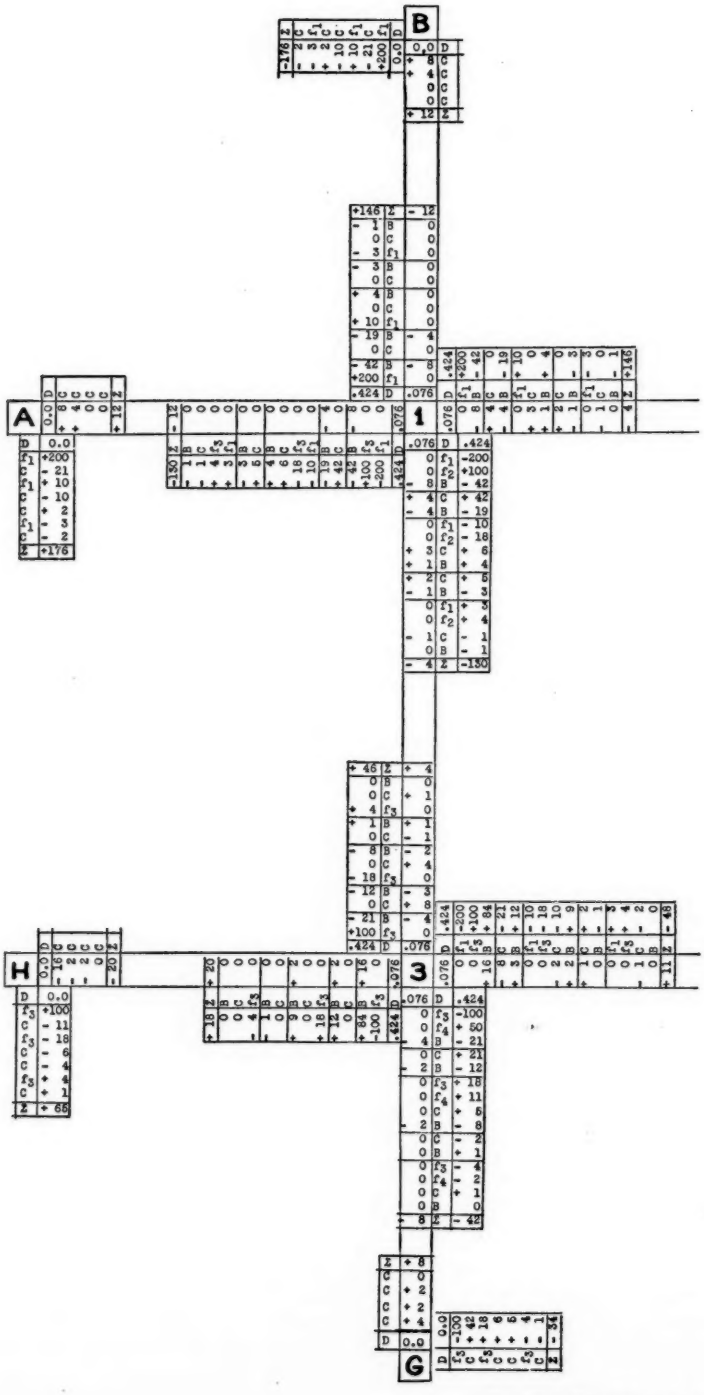
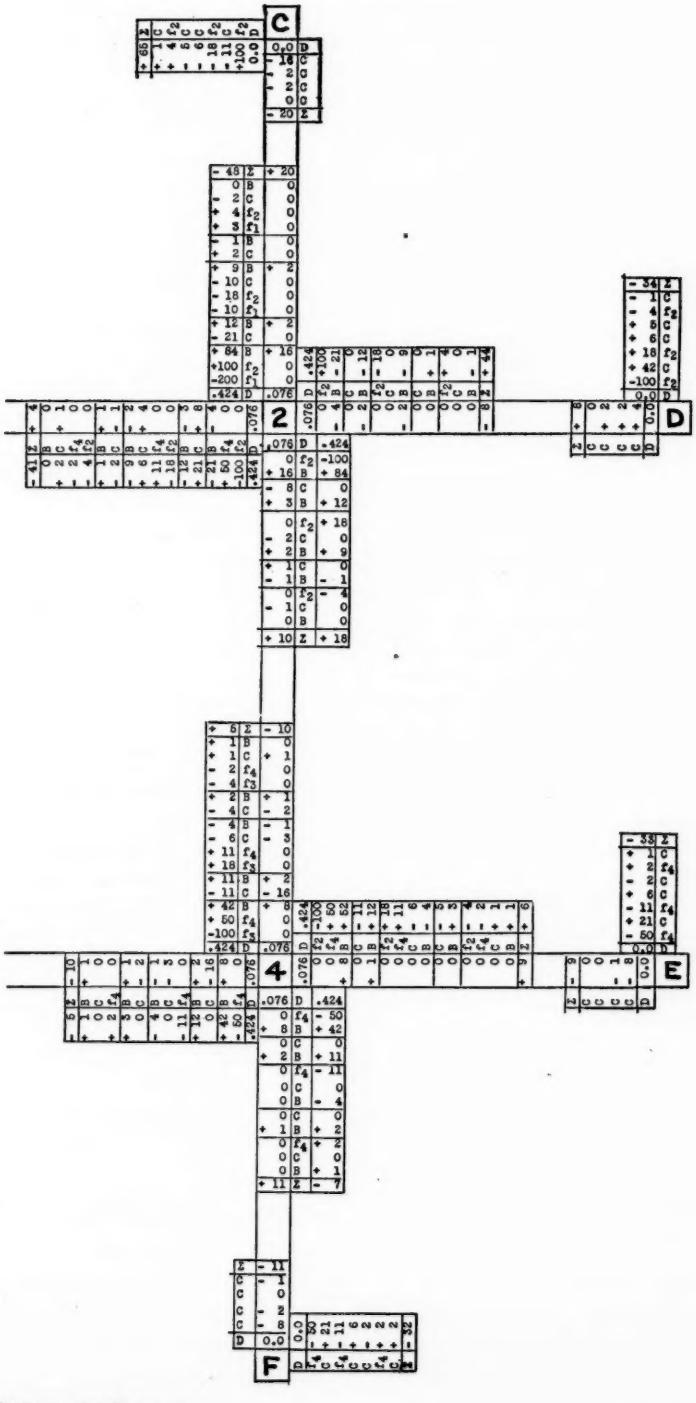


FIG. 21.—DISTRIBUTION OF



MOMENTS AND TORQUES

Since the amount of fixed-end moment introduced is a measure of the displacement given each joint, the initial values should be entered into the tabular solution in relation to the anticipated displacement of each joint. Here the analyzer must exercise his judgment and experience. The closer these initial fixed-end moments are to the total joint-displacement moments for the structure the more rapidly the solution will converge. For the gridwork shown in Fig. 20, it is apparent that joint 1 will deflect the most, joint 4 the least, and joints 2 and 3 an equal amount somewhere between 1 and 4. Therefore in the tabular solution shown in Fig. 21, as an initial estimate fixed-end moments have been introduced as follows: Joint 1, $f_1 = 200\ P\ L$; joints 2 and 3, $f_2 = f_3 = 100\ P\ L$; and joint 4, $f_4 = 50\ P\ L$.

Step 3.—The joints are unlocked and two cycles of moments and torques are distributed. Moment values are summed and substituted into the shear equations (Eqs. 23) to check the accuracy of this approximation. The following results are found for the left-hand side of the equations: Joint 1, $-\ 76\ P\ L$; joints 2 and 3, $+\ 144\ P\ L$; and joints 4, $-\ 88\ P\ L$. Since these should all be equal to zero, it is seen that joints 1 and 4 have not been displaced enough and joints 2 and 3 have been displaced too much.

Step 4.—With the joints again locked, additional fixed-end moments created by joint displacement are introduced in an attempt to satisfy the shear equations: Joint 1, $f_1 = 10\ P\ L$; joints 2 and 3, $f_2 = F_3 = -\ 18\ P\ L$; and joint $4 = 11\ P\ L$.

Step 5.—The joints are again unlocked and two cycles of moments and torque are distributed.

Summary.—The foregoing procedure is repeated until finally all moments about a joint are in balance and the shear equation for each joint has been satisfied. The structure is now in complete equilibrium. The moment totals shown in Fig. 21 indicate the true values of moments and torque at each end of the members.

To obtain the deflection of each joint, it is necessary only to sum up the fixed-end moments introduced by successive joint displacements, equate these totals to $6\ E\ I\ \Delta/L^2$, and then solve for Δ. The following results are then obtained:

$$\frac{6\ E\ I\ \Delta_1}{L^2} = (200 + 10 - 3)\ P\ L = 207\ P\ L;\ \text{and}\ \Delta_1 = \frac{34.5\ P\ L^3}{E\ I} \ldots (24a)$$

$$\frac{6\ E\ I\ \Delta_2}{L^2} = \frac{6\ E\ I\ \Delta_3}{L^2} = (100 - 18 + 4)\ P\ L = 86\ P\ L;$$

$$\text{and}\ \Delta_2 = \Delta_3 = \frac{14.4\ P\ L^3}{E\ I} \ldots (24b)$$

and

$$\frac{6\ E\ I\ \Delta_4}{L^2} = (50 + 11 - 2)\ P\ L = 59\ P\ L;\ \text{and}\ \Delta_4 = \frac{9.9\ P\ L^3}{E\ I} \ldots (24c)$$

The values of Δ in Eqs. 24, when divided by 1,000, compare favorably with the results obtained by solving Eqs. 16. More accuracy could be obtained by increasing the cycles of distribution and by using more significant figures.

The foregoing method of successive shear corrections has some advantages and some disadvantages. The advantages are as follows:

1. A single solution is required no matter how many joints are involved.
2. To find the deflections no simultaneous equations are needed at the end of the solution. This becomes important when the number of joints involved is great.
3. Final answers for torques and moments are obtained at the end of a single solution.
4. The analyzer can use his judgment and experience in estimating the relationships between the various joint displacements, thus speeding the solution.
5. The method of first using a coarse net, and then with the displacements obtained therefrom proceeding to a finer net, can be pursued.
6. In cases of symmetry of loading it is possible to use short cuts similar to those used by the authors.

The disadvantages are as follows:

a. In certain cases the single solution mentioned as advantage 1 is somewhat cumbersome; and
b. Each new loading condition on the structure involves a separate solution.

In any particular problem the analyzer must decide whether to use the superposition method or the successive shear corrections method. Each has a definite advantage in certain types of problems, and the analyzer should be familiar with both.

PAUL J. BRENNAN,[15] J. M. ASCE.—Evidence on the action of reinforced concrete slabs is obtained from analysis, testing, experience, and judgment. Information from one source may substantiate evidence from other sources, may disagree with evidence from other sources, or may introduce evidence that is not apparent in other sources. Nonetheless, all are necessary for an understanding of the structural action of reinforced concrete slabs.

Mathematical analysis is an important and relatively inexpensive source of evidence and a necessity in the interpretation of data obtained from tests. The absence of a satisfactory method of analysis is one of the major impediments in the study of slabs. At present, slab analysis is restricted to the mathematical theory of elasticity, methods of static control, approximate formulas, and systems that assume the slab to be a grid. Each approach is limited.

The mathematical theory of elasticity is limited to homogeneous and isotropic plates. The expression defining the deflections must be known, and must satisfy the boundary conditions and the Lagrange equation. The Lagrange expression, which equates the statics to the slab deformations through the properties of the material, is not general. It is only applicable when the moduli of elasticity are the same in all directions and the moments of inertia about horizontal axes are equal in all directions. Reinforced concrete slab construction does not lend itself to these conditions.

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The static control approach is powerful and exact for some conditions. Although the total moment across any section or sections of a slab may be known, the manner in which these moments are distributed is not known. This may not be serious, however, because, if provision is made to resist this total moment, the slab will probably adjust automatically to resist it and there is reason to believe that it does.

The general grid approach has the advantage of showing a physical picture. It is somewhat straightforward and uses tools that have become familar to many engineers. However, it does have disadvantages. When a distribution procedure is used in some methods convergence is not automatic. The number of strips to be considered is not obvious. Dependence is placed on values of torsional resistance in concrete which are nebulous, but the lack of such values is probably not serious.

The grid analysis system explained in the paper is one of a number of systems that may be developed. For example, a system using a grid of beams dependent upon linear simultaneous equations for the solution of deflections was presented in 1933 by John N. Pirok.[16] The method presented by Messrs. Ewell, et al., has the disadvantage that it is dependent upon the solution of linear simultaneous equations. The grid system that is dependent, in whole or in part, on linear simultaneous equations is open to discussion. It is possible to solve a few equations containing a few unknowns if the equations are of the right form. Symmetrical slabs with symmetrical loadings do not necessarily involve a prohibitive number of equations. The unsymmetrical slab with an unsymmetrical load is common in structures and by the system described involves one equation and one unknown for each grid intersection. The satisfactory solution of these equations for a sufficient number of grid beams to represent this slab is questionable.

The writer believes that a grid system approach to the analysis of reinforced concrete slabs is valuable and may be an important source of evidence in the study of slabs. However, it must be applicable to any slab, with any variation in thickness, any support conditions, and any loading. The mathematical approach is limited by the assumptions on which it is based. If the reinforced concrete slab does not conform to the assumptions, the mathematical analysis may be an unsatisfactory source of evidence. Even with reasonable conformation between the assumed action and the structural action, mathematical analysis is not sufficient by itself. Testing, experience, and judgment must be considered and evaluated with analysis.

SERGE LELIAVSKY BEY,[17] M. ASCE.—Not only does this paper have outstanding merit, but also much timeliness, for the problem it deals with—namely, the bearing capacity of a thin plate supported (or fixed) on four or two sides—calls for further and possibly deeper study, which, the writer believes, may result (if successful) in a material saving in the expenditure

[16] "Studies in the Analysis of Slabs," by John N. Pirok, thesis presented to the University of Illinois, at Urbana, Ill., in 1933, in partial fulfilment of the requirements for the degree of Master of Science in Civil Engineering.

[17] Director, Designing Service, Reservoirs & Nile Barrages Dept., Ministry of Public Works, Cairo, Egypt.

on steel gates and various other works. It should, nevertheless, be realized that the principle of substituting a grid for a slab is not as obvious, as it may possibly appear at first glance, for from the standpoint of elastic deformation analysis, a set of parallel beams does not deflect in the same manner as a plate —even if the mean second moments, per unit width, are the same in both cases —because of the lateral elastic constraint in the plate.

In Fig. 22 the elastic deformation is intentionally exaggerated, and the plate is assumed to be supported on two sides only, that is, AB and CD. As the plate bends, compression is developed in the upper part of the central

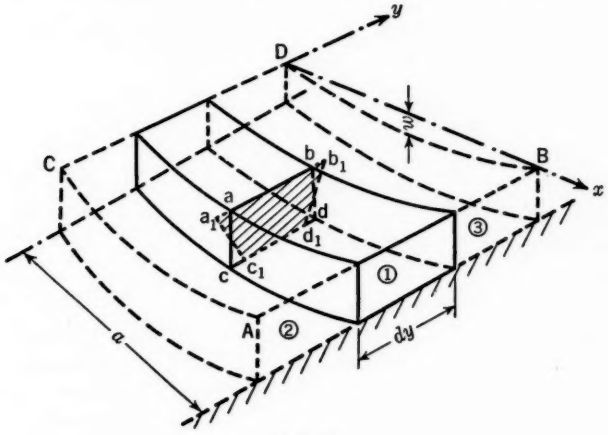

FIG. 22

section, abcd, and tension is developed in the corresponding lower part. Considering element 1 as a free body, its top width ab becomes a_1b_1 because of the lateral effect of the main stress while cd deforms to the length, c_1d_1.

However, since elements 2 and 3, adjacent to element 1, prevent these lateral deformations from taking place, a secondary compressive stress must develop, at right angles to the main, primary bending stress in the upper part of the plate; also, a secondary tensile stress must be found to occur in the lower part of the same section. These additional stresses can be visualized as being caused by a constraint moment, acting at right angles to the main bending moment—very much as if one side of the plate had been heated while, at the same time, deformations were prevented.

To estimate quantitatively the importance of this lateral effect, the general equation of the elastic plate

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{N} \qquad (25)$$

may be employed. The symbol w represents the vertical deflection, whereas N is the so-called "rigidity factor"

$$N = \frac{Eh^3}{12(1-\gamma^2)}$$

in which E and γ are, respectively, the Young modulus and the Poisson ratio, and h is the thickness of the plate. Since in Fig. 22 w is constant for given values of x, Eq. 25 becomes[18]

$$\frac{\partial^4 w}{\partial x^4} = \frac{p\,f(x)}{N} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Assuming that p is constant, integrating, and substituting $w = 0$ and $M_x = 0$, for either $x = 0$ or $x = a$,

$$w = \frac{p}{24\,N}\left(x^4 - 2\,a\,x^3 + a^3\,x \right) \ldots\ldots\ldots\ldots\ldots\ldots (27a)$$

as against

$$w = \frac{p}{2\,E\,h^3}\left(x^4 - 2\,a\,x^3 + a^3\,x \right) \ldots\ldots\ldots\ldots\ldots\ldots (27b)$$

for the case of ordinary beams such as those of the grid. Comparing the coefficients in Eqs. 27, it follows that Eq. 27*a* can be derived from Eq. 27*b* by multiplying it by the factor $(1 - \mu^2)$.

Assuming that for steel $\mu = 0.3$, it follows that the deflections of an equivalent grid are about 10% greater than those of a plate. Thus, the plate is 10% more rigid than the grid, owing to the effect of the lateral constraint. The authors' comments on this point can enhance the value of the paper. In particular since the deflections calculated by the authors' method were always greater than those computed by Eq. 19 (see Table 3).

In most engineering structures such as steel gates or reinforced concrete decks, the designer is not concerned with a single unit, but with a group of contiguous plates. This fact introduces a new element in the mechanical analysis of the problem—namely, the horizontal component, S, of the reaction. Had the plate been ideally rigid or, alternatively, had its edges been provided with sliding or roller bearings, this force could not have developed. However, since the plate is usually even more flexible than a beam, and its edges are either cast together, riveted, or welded to the supporting beams (which possess a certain degree of lateral rigidity, or are prevented from moving transversly by the plates spanning the adjacent panels), it follows that each individual panel will work as a chain or rope, somewhat in the same manner as a suspension bridge.

To the usual equation of the elastic axis of the plate—

$$E\,I\,(1 + \mu^2)\,\frac{d^2 w}{dx^2} = \frac{p\,a^2}{4} - \frac{p\,x^2}{2} \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

—another term, $-S\,w$, must, therefore, be added; and, thus,

$$E\,I\,(1 + \mu^2)\,\frac{d^2 w}{dx^2} = \frac{p\,a^2}{4} - \frac{p\,x^2}{2} - S\,w \ldots\ldots\ldots\ldots\ldots (29)$$

[18] "Elastische Platten," by A. Nadai, J. Springer, Berlin, Germany, 1925, p. 70.

Integrating Eq. 29, Ph. Forchheimer[19] obtains

$$w = \frac{P}{S\,r^2}\left[1 - \frac{\cosh\,(r\,x)}{\cosh\left(\frac{r\,a}{2}\right)}\right] - \frac{p\left(\frac{a^2}{4} - x^2\right)}{2\,S} \quad \ldots\ldots\ldots\ldots (30)$$

in which

$$r = \sqrt{\frac{S}{E\,I\,(1+\mu^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

For the maximum deflection, in the middle of the span, Eq. 30 yields

$$w_0 = \frac{p}{S\,r^2}\left[1 - \operatorname{sech}\left(\frac{r\,a}{2}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

and the corresponding moment, obviously, is equal to

$$M_0 = \frac{p\,a^2}{8} - S\,w_0 = \frac{p}{r^2}\left[1 - \operatorname{sech}\left(\frac{r\,a}{2}\right)\right] \quad \ldots\ldots\ldots\ldots (33)$$

The problem is thus solved, provided S is known. In order to calculate this force, the length, β, of the elastic axis must be known; then

$$S = \frac{\beta - a}{a}\,E\,(1+\mu^2)\,h \quad \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

It is well known that analytical determination of the lengths of arcs is frequently a difficult problem. In the present case Forchheimer introduced certain simplifications, and thus obtained a solution that can be presented in the following form:

$$\frac{16\,h^4\,E\,(1+\mu^2)}{p\,a^4} = f\left(\frac{r\,a}{2}\right) = \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

For the function $f(r\,a/2)$ he produced a table which, for practical convenience, the writer has converted into a graphical form as shown in Figs. 23 and 24, depending upon whether the plate is freely supported or fixed. In the latter case, that is, for a fixed plate, the formula for the bending moment will include an additional term, the fixation moment M_f, and the symbol ψ will be used to denote the relevant function (in place of ϕ).

For practical applications of the grid analogy, forming the subject matter of the paper under discussion, it is rather important to consider the force, S, for in this manner a solution may possibly be obtained, explaining—at least partly—the fact that, in nature, plates are capable of withstanding far greater loads than those calculated from the various elastic bending theories.

To afford an idea of the exceedingly high capacity of elastic plates in nature it will suffice to quote some of the classical experiments of C. Bach,

[19] "Die Berechnung ebener und geecrümmter Behälterboden," by Ph. Forchheimer, W. Ernst, Berlin, Germany, 1909, p. 3.

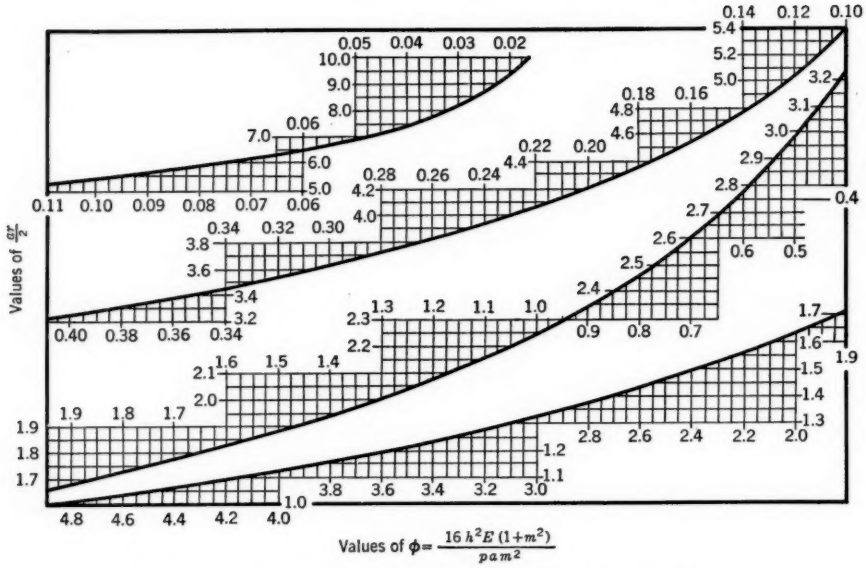

FIG. 23.—FORCHHEIMER'S FUNCTION ψ, FOR PLATES FREELY SUPPORTED

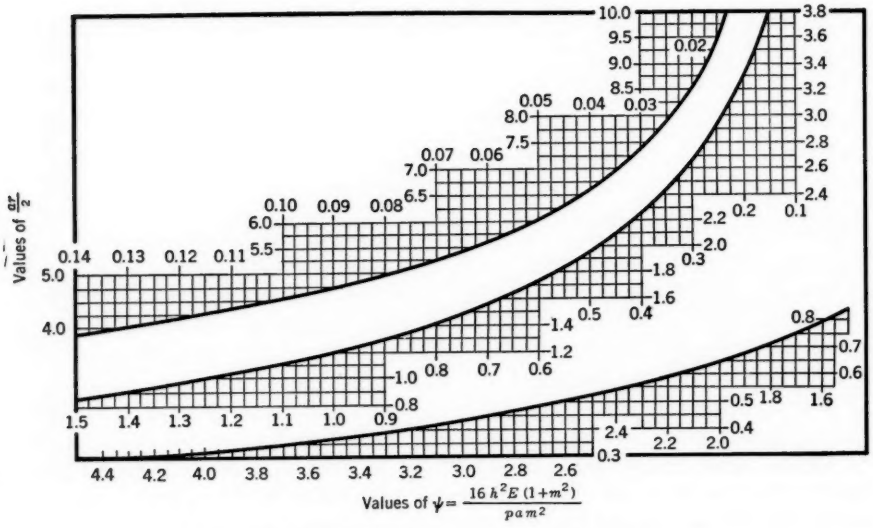

FIG. 24.—FORCHHEIMER'S FUNCTION ψ, FOR FIXED PLATES

which are mentioned in so many reference books (without, however, giving the actual test data):

a. Square Plate.—Sides $a = b = 80$ cm, thickness $h = 0.8$ cm; computed permissible unit load $p = 0.48$ atmosphere. The pressure was raised to 24 atmospheres without failure, or any sign of failure.

b. Square Plate, Same Size As Item a But with $h = 1.6$ Cm.—Calculated $p = 1.92$ atmospheres. The pressure was raised to 20 atmospheres. There was no sign of failure, but the test was discontinued because of leakage in the joints.

c. Rectangular Plate.—$a = 40$ cm and $b = 80$ cm, $h = 0.8$ cm; computed load $p = 1.2$ atmospheres; pressure raised to 28 atmospheres. No sign of failure.

d. Rectangular Plate, Same Size As Item c But with $h = 1.6$ cm.—Calculated $p = 4.8$ atmospheres; actual $p = 34$ atmospheres, after which the test was discontinued, because of leakage in the joints.

It appears from these data that here are phenomena belonging to the plastic range of conditions. It may be of interest to mention in this connection that, according to R. N. Southwell, the relaxation method, employed by the authors, is particularly suited for dealing with such cases, and it would therefore be of considerable interest to develop the solution in this direction.

To sum up this discussion, there are three points that require further explanation or development—(*a*) the effect of the $(1 + \mu^2)$-factor, (*b*) the effect of the S-force, and (*c*) the effect of plasticity.

W. Watters Pagon,[20] M. ASCE.—There have been solutions of this problem for slabs of circular and square shape, but when the shape becomes irregular exact analysis has heretofore been impossible. In general, it remains so. Of course, following the authors' method, the work involved becomes stupendous, unless the computations can be so arranged as to be handled by a mechanical computer.

A trapezoid, such as that in a skew span, has been treated fairly extensively by others, but in modern concrete bridges there often arises the problem of stress in an irregularly shaped quadrilateral such as the type occurring at the entrance or exit of a ramp approach. Such a slab at times also requires a central "edge beam" extending from a salient corner, which makes the problem essentially one of two adjacent, continuous spans straddling the central edge beam with negative moment over it, and all complicated by the irregular shape. A very flat dome roof, with reinforcing ribs and local concentrations, in addition to the customary uniform load causing radial and circumferential stress, might well be analyzed by the authors' method.

Specifications of the American Association of State Highway Officials (AASHO), for the design of an edge beam, are a compromise, and permit the design of a deep curb beam or the addition of extra reinforcement to the outer edge of the slab. This is a problem which in the writer's opinion deserves

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re-analysis, and might perhaps be done by the authors' method. Of course, the writer fully realizes that these suggestions contemplate extensive work, and the use of mechanical or electrical computers; however, it does not seem that there is any other recourse to some of the problems other than by using expensive and elaborate model tests.

The authors are to be congratulated on their efforts.

PHIL M. FERGUSON,[21] M. ASCE.—The possibility of analyzing slabs by procedures with which many structural engineers are already familiar is introduced in this well written paper. It is true that other methods of slab analysis, involving difficult mathematics, have made great progress and have been the basis for productive numerical methods. Nevertheless, the field of such methods thus far has been limited to relatively simple boundary conditions. This new approach appears to be on a very flexible basis; even variable sections and slab openings are minor rather than major difficulties.

This is an entirely new approach in the sense that related earlier work appears not to have found its way into English publications. Since foreign language reports are not actually available to most American engineers, those who report promising developments from abroad deserve recognition. In this case the authors have gone further and introduced American procedures that are a distinct improvement.

The possibilities for this method are many, but several factors need further appraisal. The first factor is the accuracy of the results. Deflections of slabs are of secondary importance structurally compared to bending moments in slabs. Methods that are adequate for deflections may not be entirely adequate for establishing satisfactory moments. The authors have checked the accuracy of their methods and the results shown in Table 3 are quite good. Nevertheless, other comparisons would add much to the paper. It is hoped that their closing discussion will show such comparisons, which they report as available, and will also include a brief restatement of Béla Enyedi's numerical comparison.[6] The writer would like to have this additional basis for evaluating the general accuracy and usefulness of the method.

In addition to the approximations involved in using finite widths of strips (and relatively large finite widths) instead of differential quantities, it appears that this procedure now neglects the Poisson ratio effect. Possibly this is one reason the authors' values in Table 3 are slightly too large. (Another factor is discussed later that should make these calculated values too small; the two effects partly offset each other.) It appears that it might be necessary to incorporate Poisson's ratio in some way before really good values of moment would be obtained. The writer does not quite visualize the manner in which this can be done, but believes that it is possible, certainly as an approximation if not more accurately.

The method as presented is limited by the fundamental difficulty of handling a large number of simultaneous equations. Before much further progress can be made, moment distribution methods using successive approximations must

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[6] "Trägerrost und Platte," by Béla Enyedi, *Beton und Eisen*, Vol. 34, Heft 3, 1935, p. 42.

be made available for this type of problem. At present such methods for extended grids are themselves difficult. The writer has used successive approximations to solve grids slightly more complex and more extended than the twleve-joint quadrant used in Example B; but such solutions can scarcely be classified as practical. If grid methods can also be used for slab analysis, as this paper suggests, further study of such procedures is quite important.

The writer would like to recommend the tabulation form used in Fig. 6 as being far superior to that used in Fig. 11. It is realized that Fig. 11 covers rotations about both axis xx and axis yy. These calcuations would be easier to follow if separate tabulations were set up for these two axes. For members intersecting at right angles there is no transfer of data between the calculations for the two axes; hence, tabulations need not be adjacent.

In one respect the writer is critical of the authors' detailed procedure. Figs. 9 and 13 show slabs with fixed edges. The gridwork substituted for these slabs ignores a half strip width adjacent to each fixed edge. Since these half strips also deform, their neglect seems equivalent to assuming a fixed edge one half strip inside the ones originally shown in the statement of the problem. All the authors' calculated deflections should thus be too small. Table 3 indicates otherwise, which might seem to refute this argument. However, the writer believes that this is simply evidence that some other important factor giving an opposite effect has been omitted, possibly Poisson's ratio.

Further study of this method is definitely needed before it can be fully accepted. However, the method has such promise (if it can be entirely validated) that such further study is more than justified.

The writer knows of no other method that approaches this close to the solution of practical problems: (*a*) Of flat slabs that are not entirely regular, (*b*) of slab band construction, or (*c*) of slabs with holes cut through them—in other words, the design of slabs as the structural engineer encounters them, day in and day out. Even if the method cannot be fully validated, it may yield relative values that will be significant and illuminating. Hence, he predicts that much more will be heard of this method in the future. The authors are to be commended for this pioneering step.

ROBERT V. WHITMAN,[22] J. M. ASCE.—This discussion is concerned with the authors' treatment of the so-called grid analogy for obtaining solutions to problems in the bending of flat plates. The fundamental concept behind this analogy is certainly sound. However, the paper leaves in doubt several points concerning the stiffness factors to be applied to the beams of the grid.

First, consider the choice of the torsional stiffness factor. In both of the plate examples treated by the authors, the aspect ratio of the grid beams was calculated and the corresponding value of the factor β in Eq. 3*b* was then selected. Suppose that the number of the beams in the grid were to be increased many times, the plate thickness remaining unchanged. The factor β would decrease rapidly, and correspondingly the relative importance of the torsional stiffness factor would increase as compared to the bending stiffness factor. In the limiting case of infinitely many very thin beams, β would vanish and the calculation would indicate no twisting moments in the plate.

[22] Research Associate, Dept. of Civ. and San. Eng., Massachusetts Inst. of Technology, Cambridge, Mass.

This consideration indicates that the use of a torsional stiffness factor coupled to the aspect ratio of the grid beams is not correct. Rather, the value $\beta = \frac{1}{3}$ should be used at all times. The results of the calculations can then be used for a plate of any thickness, assuming of course that the limitation on the magnitude of the transverse shear forces is observed. Following the authors' line of reasoning, the deflection values listed in Tables 1 and 3 would hold accurately only for one value of plate thickness, and small changes in the distribution factors would lead to slightly different deflections for other thicknesses. The values of β used by the authors were very nearly one third, and thus these comments affect the general theory rather than their numerical results.

The choice of the bending stiffness factor is also open to question. The authors use a factor that is independent of Poisson's ratio, although it is well known that the bending stiffness of the beam elements of a plate is increased in the ratio $\frac{1}{1-\mu^2}$ since these elements are restrained against lateral change in thickness. It would seem that the grid analogy would be most accurate if the beams of the grid were to conform as nearly as possible to elements cut from a plate, and that the use of the increased stiffness would hence be more logical. Unfortunately, no simple, concise argument is at hand to aid in a definite choice of the proper factor.

If the suggestions in the preceding paragraphs are followed, the distribution factors appearing in Eqs. 18*a* and 18*b* will become

$$\text{Percentage moment} = 12.62 \text{ torque}$$
$$\text{Percentage moment} = 37.38 \text{ bending.}$$

A similar change would be made for the values shown in Fig. 10. These changes are not large, and the auxiliary forces listed in Table 2, for example, would not be greatly altered. Since small changes in the coefficients of simultaneous equations sometimes cause considerable change in the solutions of these equations, it is difficult to estimate how much the values in the bottom line of Table 2 will change. However, the formula used to convert the values in this line into the deflection values in Table 3 will contain the factor $\frac{1}{1-\mu^2}$, and here the effect will definitely be noticeable.

Positive answers to these questions can be obtained only from a mathematical demonstration of the validity of the grid analogy. Such a task was undertaken by Milton Platt[23] who wrote the equilibrium and continuity equations for the joints and beams of the grid system, and then applied difference and summation procedures to combinations of these equations. In the limiting case of infinitely many beams these operations yielded the differential equation for the bending of plates, thus establishing the analogy. Mr. Platt's work confirms the use of the same torsional stiffness factor for all grid beam proportions. Since he applied the demonstration only for the case $\mu = 0$,

[23] "A Network Procedure for the Analysis of Flat Plates Subjected to Transverse Distributed Loads," by Milton M. Platt, thesis presented in 1946, to the Massachusetts Inst. of Technology, Cambridge, Mass., in partial fulfilment of the requirements for the degree of Doctor of Science.

further study is necessary to establish definitely the proper bending stiffness factor.

As stated by the authors, the labor required to solve plate problems in this fashion is still considerable. As a result of some experience with the procedure, it is estimated that an experienced person using this procedure on a plate such as the one in Example C would consume about 40 hr for the determination of the auxiliary force system, and another 10 hr to solve and check the simultaneous equations and to assemble the desired results. By far the major part of the work is in the moment distribution steps. The authors suggest one

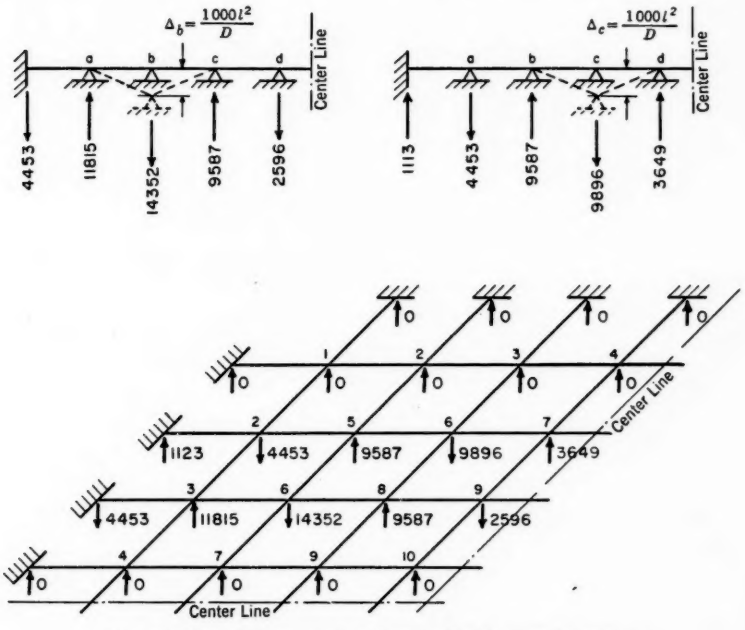

FIG. 25.—COUNTERPART OF FIG. 16 WITH NO TORSION INCLUDED

possibility for reducing this numerical effort, that is, using a limited number of cycles of distribution. Another approximation can be obtained by neglecting the torsion of the grid beams. Then beams intersecting at a joint will have the same deflection but not necessarily the same slopes. It is instructive to see the magnitude of the errors that may result from such an assumption.

When there is no torsion it is only necessary to apply conventional moment distribution procedures to isolated beams. For example, consider the plate in Example C. Fig. 25 indicates how the auxiliary forces caused by deflection Δ_6 can be obtained by superimposing the results of distributions on only two beams. With but two additional distributions a table corresponding to Table 2 can be prepared. The work of forming the table has been reduced, and it will contain a good number of zeros. Such a "no-torsion" solution has been made assuming the stiffness of the grid beams to be proportional to $\frac{EI}{1-\mu^2}$, and

TABLE 4.—COMPARISON OF MAXIMUM DEFLECTIONS AND MOMENTS FOR UNIFORMLY LOADED, SQUARE PLATES (See Fig. 13)

Method of Computation	(a) FOUR EDGES CLAMPED (EXAMPLE C)			(b) TWO ADJACENT EDGES	
	Deflection	Positive moment	Negative moment	Deflection, point 10	Moment, line D
(1)	(2)	(3)	(4)	(5)	(6)
I. A. Wojtaszak[a]	$0.1308 \frac{w a^4}{E h^3}$	$0.0233\ w\ a^2$	$0.0513\ w\ a^2$	....	....
Grid Analogy—Authors'	....	....	....	$0.292 \frac{w a^4}{E h^3}$	$0.241\ w\ a^2$
No torsion	$0.01647 \frac{w a^4}{E h^3}$	$0.0233\ w\ a^2$	$0.0542\ w\ a^2$	$0.705 \frac{w a^4}{E h^3}$	$0.334\ w\ a^2$
Percentage error	26	0	6	142[b]	39[b]

[a] *Transactions*, A.S.M.E., Vol. 59, 1937, p. A. 173. [b] Percentage error with no torsion.

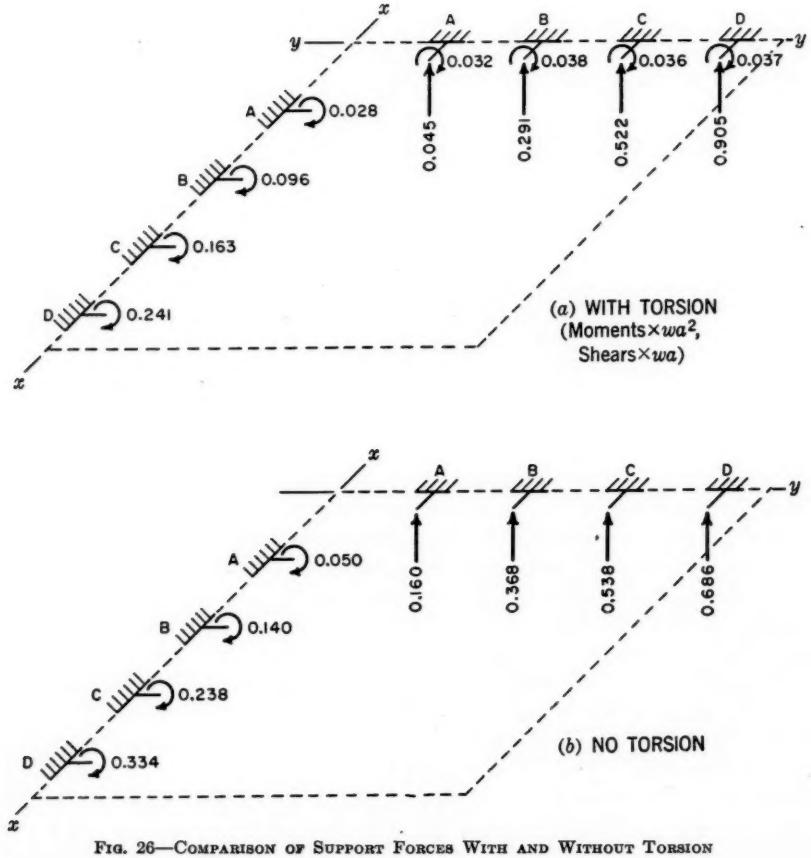

FIG. 26—COMPARISON OF SUPPORT FORCES WITH AND WITHOUT TORSION

the solution has been summarized in Table 4(*a*). Although the error in the maximum deflection is considerable, the agreement in the moments is remarkable.

As a second example of the importance of torsional forces, consider the case of a square plate with sides of length *a*, with two adjacent edges clamped and the other two edges completely free, and subject to a uniformly distributed load. All other properties are the same as for the plate of Example C. The manner of choosing grid beams is the same as for one quadrant of that plate. Except for the distribution factors used at joints 4, 7, 9, and 10 (Fig. 13) all numerical work follows exactly the same form. Table 4(*b*) and Fig. 26 show the results of applying the grid analogy to this problem, first with torsion included and then with no torsion. The edge shears shown do not include the effect of the loads on the grid beams of width $\frac{1}{2}\,l$ which lie immediately adjacent to the fixed supports, and which do not enter into the distributions.

In this case the assumption of no torsion gives deflections which are very much in error, and the discrepancies in the moments and shears are also considerable. These larger errors are not surprising since the case is one in which torsional couples are certain to play an important part.

It is clear that the approximation of no torsional forces must be used cautiously, and only after the investigation of numerous other cases. For the engineer who is primarily concerned with obtaining estimates of the maximum stresses it may prove to be a useful tool.

Finally it is of interest to observe the accuracy of the authors' method when applied to this last example (Table 4(*b*)). When moment equilibrium about the side x–x is investigated, it is found that the various edge forces contribute to resisting the moment of the applied load in the following amounts: Bending moments along x–x, 26%; shear forces along y–y, 67%; and twisting moments along y–y, 7%. Since, theoretically, there should be no twisting moments along the clamped edges, 7% is a fair estimate of the over-all accuracy of the external forces.

WALTER W. EWELL,[24] J.M. ASCE, SHIGEO OKUBO,[25] A.M. ASCE, AND JOEL I. ABRAMS,[26] J.M. ASCE.—The writers are gratified by the interest shown in their paper and would like to thank the discussers for their worthwhile contributions.

Mr. Brennan emphasizes the importance of correlating experience and judgment with mathematical analyses of reinforced concrete slabs; with this point, the writers are in complete agreement. This is particularly true when the mathematical analysis is to be predicated on a series of ideal conditions rather than on those which actually exist. However, analysis must be considered a guide to engineering judgment and in many cases more than one type of analysis can and should be carried out. It is with this fact in mind that the grid method is presented, as an analysis to augment rather than to supplant other well-known methods.

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[25] Asst. Prof. of Civ. Eng., Univ. of Hawaii, Honolulu, T. H.
[26] Instr., Civ. Eng., The Johns Hopkins Univ., Baltimore, Md.

It is believed that the grid method of analysis can be readily applied to slabs characterized by nonisotropic materials and varying thickness. Important examples of this type of irregularity have been discussed by Mr. Pagon, who emphasizes the need for re-examination of the stress problem in an irregularly-shaped quadrilateral slab complicated by a central "edge beam," and in a flat roof dome reinforced by ribs. A grid analysis of a geometrically irregular slab is naturally more difficult because of the increase in the number of linear simultaneous equations required for a solution. Still, an approximate solution of a complicated problem is valuable. This can be effected by using a smaller number of grid beams and thereby reducing the number of equations to be solved simultaneously. Accuracy would be sacrificed for the sake of obtaining an answer that would at least indicate the structural behavior of the system.

The distribution procedures used in this paper, although not necessarily characterized by rapid convergence, can be extended to any degree of finality that is desired. This characteristic is common to most of the numerical procedures used in engineering practice. The analysis of a slab that is unsymmetrical in loading and geometry requires the solution of a large number of equations, as has been stated by both Mr. Ferguson and Mr. Brennan. However, if a problem warrants a solution at all, and if a solution cannot be obtained in another manner, this should not be considered a consequential deterrent. Mr. Pagon wisely suggests the use of electronic computing devices when such problems are encountered.

The accuracy desired in the solution determines the number of grid strips to be selected. In the limiting case of infinitely many grid strips, Mr. Platt[23] has established the fact that a grid system acts as a slab. The writers reluctantly admit that no attempt has yet been made to correlate the variation in accuracy with the variation in the number of grid strips, but future studies should provide this information. Of course, unorthodox problems must always be given individual consideration.

The writers are indebted to Mr. Scordelis for his discussion of the shear correction method as applied to the moment distribution process and his subsequent application of this method to a grid system. The procedure obviates the simultaneous solution of the shear equations. However, if the process is to be most effective, the analyst must judiciously choose a group of joint displacements that will accelerate the convergence. In a complicated slab problem, the analyst might find it difficult to select the best displacement values. Moreover, the shear correction method also becomes tedious when a large number of grid joints must be considered. In spite of these disadvantages, Mr. Scordelis' application represents an important refinement to the basic idea.

As indicated by Mr. Whitman, the effect of restraint that actually exists in any beam cut from a plate was not considered in the grid analogy. In order to make the plate and gridwork systems comparable, the restraint effect should be considered when choosing the stiffness factors. In this sense, the use of the term "analogy" is somewhat misleading.

Inasmuch as the inclusion of the restraint factors causes no new difficulty in the fundamental process, it should be done in order to make the grid conform

more closely to the plate. This important fact was emphasized by Leliavsky Bey, Mr. Ferguson, and Mr. Whitman.

The choice of restraint factors, for all cases, is still not completely clear. Mr. Whitman suggests the use of $\beta = \frac{1}{3}$ as the torsional stiffness factor for all values of the aspect ratio of the grid beam. Whether this is a valid assumption is open to discussion and further study. Mr. Whitman explains his choice by citing the case of infinitely many grid beams with slab depth held constant where, he claims, the β value vanishes. The contrary is true, however. With infinitely many grid beams,[27] the aspect ratio (depth divided by width) approaches infinity and β approaches $\frac{1}{3}$. Since all grid strips cut from the plate are restrained by adjacent material, the torsional stiffness factor should be modified. When the cross-sectional dimensions are small as compared to the length of the member, the constraint has a negligible effect upon the torsional stiffness.[28] This is true when several of the cross sections are constrained, but its effect on the case under discussion is questionable.

Leliavsky Bey and Mr. Whitman indicate that the bending stiffness should be modified by the factor $1/1\text{-}\mu^2$. The writers are especially indebted to Leliavsky Bey for his demonstration of this fact.

Leliavsky Bey attributes a plate's capacity to carry more load than that calculated by existing theory to the force in the plane of the plate, and, furthermore, presents a very practical method of calculating this force. The Forchheimer graphs (Figs. 23 and 24) greatly simplify the lateral force determination. On a gridwork of beams it is conceivable that the magnitude of axial force within any beam can be calculated by some of the better-known methods of structural analysis. In fact, Mr. Benjamin has shown how a gridwork can be used to determine the forces developed in the plane of a plate subjected to edge loading. Perhaps Mr. Benjamin's grid study can be coupled with the writers' study in order to determine axial, bending, and torsional effects simultaneously.

Mr. Ferguson has questioned the advisability of ignoring a half grid strip adjacent to the plate boundary. Still, the writers are convinced that they have used the most rational plate division. Such a plate division is not tantamount to assuming the boundary one-half grid strip away from the actual plate boundary since the perpendicular grid beams have their terminal reactions on the plate's edge and are consequently free to deflect from the edge inward. Furthermore, the slab division into equal-length grid beams reduces the computational procedure that must follow—a simplification that appears to be justified when the number of grid beams is selected so as to minimize the error.

Mr. Ferguson has requested a statement of Mr. Enyedi's comparison of moment value with that obtained by Mr. Marcus. On a simply-supported plate subjected to uniform loading with $\mu = 0.25$, Mr. Enyedi found the moment at the center to be $0.0372w\,a^2$, whereas Mr. Marcus gave a moment value equal to $0.0368w\,a^2$. For the uniformly-loaded fixed-edge plate illus-

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[27] "Theory of Elasticity," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1934, pp. 243 and 248.

[28] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., Inc., New York, N. Y., 1951, Part II, p. 282.

trated in Fig. 13, the maximum moment occurring at the midpoint of the fixed edge[29] was 15.90 w (in.-lb) whereas Mr. Timoshenko's value[30] for the same moment was 16.62 w ($\mu = 0.3$).

The writers appreciate the wealth of discussion that has been presented and recognize the fact that it has greatly enhanced the value of the original paper.

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[29] "A Grid Analogy for Analyses of Flat Plates," by Joel I. Abrams, thesis presented to The Johns Hopkins University, in Baltimore, Md., in 1950, in partial fulfilment of the requirements for the degree of Master of Science.

[30] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, p. 228.